

# Topologically Ordered Phase States: from Knots and Braids to Quantum Dimers

Luigi Martina,<sup>1</sup> Alexander Protopogenov,<sup>2,3</sup> and Valery Verbus<sup>4</sup>

<sup>1</sup> *Dipartimento di Fisica dell'Università del Salento and Sezione INFN di Lecce.*

*Via Arnesano, CP. 193 I-73100 Lecce, Italy*

<sup>2</sup>*Institute of Applied Physics of the RAS, 603950 Nizhny Novgorod, Russia*

<sup>3</sup>*Max Planck Institute for the Physics of Complex Systems, D-01187 Dresden, Germany*

<sup>4</sup>*Institute for Physics of Microstructures of the RAS, 603950 Nizhny Novgorod, Russia*

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## Abstract

We consider universal statistical properties of systems that are characterized by phase states with macroscopic degeneracy of the ground state. A possible topological order in such systems is described by non-linear discrete equations. We focus on the discrete equations which take place in the case of generalized exclusion principle statistics. We show that their exact solutions are quantum dimensions of the irreducible representations of certain quantum group. These solutions provide an example of the point where the generalized exclusion principle statistics and braid statistics meet each other. We propose a procedure to construct the quantum dimer models by means of projection of the knotted field configurations that involved braiding features of one-dimensional topology.

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## I. INTRODUCTION

The universal behaviour of low-dimensional strongly correlated systems at low temperatures is determined to a great extent by the topology of the manifolds, where the ground state and low-lying excitations are determined. In strongly correlated electron liquids with high degree of degeneracy of the ground state such a manifold is presented by a collection of strings in the form of an arbitrary tangle of knotted and linked filaments. The processes of fusion and decay of the strings give rise to modifications of the tangle and, consequently, to the universal character of the quantum criticality in phase states with topological order. Being a result of detailed study of electron liquids in the states with fractional Hall effect, this conclusion was supported recently by the results, obtained during studying dynamics of spin<sup>1,2,3</sup> and charge<sup>4,5</sup> degrees of freedom in other low-dimensional electron systems.

It is well known that statistics of excitations in  $(1+1)$ - and  $(2+1)$ -dimensional systems is connected with the braid group. Quantum states in such systems are classified by irreducible representations of the braid group instead of the even (for bosons) or odd (for fermions) irreducible representations of the permutation group as  $(3+1)$ -dimensional systems. One-dimensional irreducible representations of the braid group correspond to Abelian anyon states, while multi-dimensional irreducible representations describe the non-Abelian states of anyons. Statistics of anyon excitations is called either braid statistics or fractional statistics, because it leads to the conclusion that there exist particles with a fractional charge and spin. In the long-wavelength limit the description of non-Abelian anyons is based on the effective action of the topological field theory<sup>6</sup>, which contains the Chern-Simons term. An important particular case is the quantum group  $SU(N)_k$  case, where the integer  $k$  is the coefficient of the Chern-Simons action, or a level in the Wess-Zumino-Witten-Novikov theory.

We can also employ another approach<sup>7</sup>, based on the generalized exclusion principle. Studies of the distribution function using this approach<sup>8,9,10,11,12,13</sup> have shown that this method is equivalent to the thermodynamic Bethe ansatz<sup>14</sup>. The equation, which determines the minimum of free energy, has the form of the Hirota equation<sup>15</sup>. It is well known from the theory of nonlinear equations, that this discrete equation in the continuous limit<sup>16</sup> yields the known integrable hierarchies of nonlinear equations. An important feature of the derivation of discrete equations in the theory with the generalized exclusion principle is the absence of any reference to the dimensionality of the space, unlike braid statistics. Formally generalized

exclusion principle statistics<sup>7</sup> may take place not only in low-dimensional systems. The reason of the emergence of the solutions, which exist only in low-dimensional systems, is that in the limit of large values of momentum, discrete equations in the universal sector<sup>17</sup> of exclusion statistics encode actually the particle fusion rules, which take place in the conformal field theory. Going over to the limit  $k \gg 1$  in discrete equations of motion, the specificity of low-dimensional situation vanishes.

The features of considered statistics are summed up by the theory of tensor categories<sup>18,19</sup>. It unifies consistently the processes of braiding and fusion of string manifolds, which are images of quasiparticle world lines. In the case of low-dimensional systems this occurs in the theory of modular tensor categories<sup>18,19</sup>. The theory of symmetric tensor categories<sup>18,20,21</sup> is applicable to the  $(3+1)D$  systems. In the last case, restrictions on solutions of equations of motion are so strong, that all anyon states are excluded, and only bosons ( $\alpha = 2\pi$ ) and fermions ( $\alpha = \pi$ ) with the interchange phase  $\alpha$  remain out of anyon states.

To solve the problems of the theory of strongly correlated systems on a lattice, it is often convenient to use the theory of the braid group representation or the Temperley-Lieb algebra (TLA) representations<sup>22</sup> with a special value of the TLA parameter. In the continuous limit, we can also employ the effective Chern-Simons action<sup>23,24</sup>. In particular, to classify the hierarchy of phase states with the aid of  $(3+0)$ -dimensional spinor Ginzburg-Landau functional<sup>25</sup>, it is convenient to use the Hopf invariant<sup>26,27,28</sup>, which is the  $(3+0)D$  analog of the Chern-Simons term.

The construction of a lattice model out from the continuous theory (even inheriting its essential properties) is evidently an ambiguous procedure. Some intuitive insight of how this can be done, based on the mentioned properties, may include the following consideration. It is well known that the Chern-Simons term in the action of  $(2+1)$ -dimensional systems encodes the invariant description of fluctuating Aharonov-Bohm vortices. The action of the doubled Chern-Simons theory (for example  $SU(N)_k \times \overline{SU}(N)_k$ <sup>21,23,24</sup>) which refers to the systems, where  $T$  and  $P$  invariances are not broken, includes pairs of Aharonov-Bohm vortices with the opposite chiralities. It is natural to suppose that neutral pairs of Aharonov-Bohm vortices in such systems are plane slices of a string loop (from the three-dimensional point of view). A pair of Aharonov-Bohm vortices are portions of the loop cutted by a plane. This means on the whole that a small loop with the scale of the order of the lattice constant induces a dimerized configuration of currents when it is projected on the plane. Such a

projected loop, or equivalently the dimer configuration, can be a building block for the formation of self-organized mesoscale structures in the form of nets. The increased interest to quantum dimer distributions<sup>24,29,30</sup> is connected not only with the theory of resonance valence bonds or with the support originating from the experience of the exact solvable models<sup>31</sup>, but it is also motivated by recent results<sup>2,32</sup> in the field of non-Abelian gauge theory.

The goal of this paper is two-fold. In the second section we will discuss the opportunities, which appear due to mapping essential configurations of the (3+0)-dimensional spinor Ginzburg-Landau model into lattice  $(2+1)D$  dimer configurations. We will show that two-dimensional dimer field configurations, distributed on two sub-lattices, may be obtained by projecting three-dimensional current configurations of the so-called toroid phase state. In the third section we will consider the solutions of nonlinear discrete equations of the thermodynamic Bethe ansatz and will show their relation with the characteristics arising in the approach, based on the use of braid statistics. They are characterised by the quantum dimension, which is obtained as the solution of the mentioned discrete equations. We will give also some arguments in favor of stability of arising mappings of string nets, built of golden chains<sup>33</sup>.

## II. DIMER CONFIGURATIONS OF VORTEX PAIRS

In this section we show how to construct dimer distributions starting from Hopf links. Indeed, knots and links of field configurations appear naturally in the long-wavelength description of  $(3+0)D$  systems with spin as follows. Let us consider a gauged Ginzburg-Landau model for the charged two-components order parameter  $\Phi = (\phi_1, \phi_2)^T$  given by the free energy functional

$$F = \int d^3x \left[ \sum_{\alpha=1}^2 \frac{1}{2m_{\alpha}} \left| \left( \hbar \partial_k + i \frac{2e}{c} A_k \right) \phi_{\alpha} \right|^2 + \frac{(rot \mathbf{A})^2}{8\pi} + V(\Phi) \right], \quad (1)$$

with a generic form of the potential  $V(\phi_1, \phi_2)$ <sup>28</sup> and interacting with an internal vector gauge potential  $\mathbf{A}$ .

The components of the order parameter can be interpreted as the hopping and pairing amplitudes, respectively, in a  $t - J$  model<sup>34</sup>. The relations between these amplitudes can be

reshaped in terms of the function  $\rho$  and a real unit 3-vector  $\mathbf{n}$  defined by

$$\rho^2 = \sum_{\alpha=1}^2 \frac{|\phi_\alpha|^2}{2m_\alpha}, \quad n^a = \frac{\overline{\phi_\alpha} \sigma_{\alpha\beta}^a \phi_\beta}{2\sqrt{m_\alpha m_\beta}}, \quad (2)$$

where  $\sigma^a$ ,  $a = 1, 2, 3$  are the Pauli matrices. A residual phase-like degree of freedom is encoded into a momentum contribution to the effective gauge field

$$\mathbf{c} = \mathbf{a} - \mathbf{A}, \quad \mathbf{a} = \frac{1}{\rho^2} \sum_{\alpha=1}^2 \frac{i}{2m_\alpha} [\phi_\alpha \nabla \phi_\alpha^* - c.c.]. \quad (3)$$

Thus, one can map the model (1) into the extended version of the  $O(3)$  nonlinear  $\sigma$  model<sup>25</sup>

$$F = F_{\mathbf{n}} + F_\rho + F_{\mathbf{c}} + F_{\text{int}} = \int d^3x \left[ \frac{1}{4} \rho^2 (\partial_k \mathbf{n})^2 + (\partial_k \rho)^2 + \frac{1}{16} \rho^2 \mathbf{c}^2 + (F_{ik} - H_{ik})^2 + V(\rho, n_1, n_3) \right], \quad (4)$$

where, in dimensionless variables, the field strength  $F_{ik} = \partial_i c_k - \partial_k c_i$  and the Mermin-Ho vorticity  $H_{ik} = \partial_i a_k - \partial_k a_i = \mathbf{n} \cdot [\partial_i \mathbf{n} \times \partial_k \mathbf{n}]$  have been introduced.

Taking into account only homogeneous density states (i.e.  $\rho = \text{const}$ ) free energy is bounded from below by the inequality<sup>27</sup>

$$F_{\mathbf{n}} + F_{\mathbf{c}} + F_{\text{int}} \geq 32\pi^2 |Q|^{3/4} (1 - |L|/|Q|)^2, \quad (5)$$

given in terms of the entries of the symmetric matrix<sup>28</sup>

$$K_{\alpha\beta} = \frac{1}{16\pi^2} \int_{\mathcal{M}} d^3x \varepsilon_{ikl} a_i^\alpha \partial_k a_l^\beta = \begin{pmatrix} Q & L \\ L & Q' \end{pmatrix}, \text{ where } a_i^1 \equiv a_i, \ a_i^2 \equiv c_i. \quad (6)$$

Taking the boundary condition  $\mathbf{n}(\infty) \rightarrow (0, 0, 1)$ , one compactifies  $\mathbb{R}^3 \rightarrow S^3$  and  $Q$  is the degree of the mapping  $\mathbf{n} : S^3 \rightarrow S^2$ . That is,  $Q$  measures the linking and knotting of the filaments, which are pre-images in  $S^3$  of an arbitrary chosen value  $\mathbf{n}(x, y, z) = \mathbf{n}_0 \in S^2$ . General arguments of the homotopic group theory<sup>35</sup> ensure that  $\pi_3(S^2) = \mathbb{Z}$ , i.e.  $Q \in \mathbb{Z}$ , labelling distinct sectors among all configurations of the field  $\mathbf{n}$ . For two linked loops one has  $Q = 1$ , for the trefoil knot  $Q = 6$  and etc. On the other hand,  $L$  is the mutual linking number<sup>35</sup> of the fields  $\mathbf{c}$  and  $\mathbf{a}$ . The integer  $Q$  is related<sup>28</sup> to the coefficient  $k$  in the Chern-Simons action. For instance, the value  $Q = 1$  for the Hopf link of two loops is equivalent to  $k = 2$ .

But, differently from  $\mathbf{n}$ , the length of the vector  $\mathbf{c}$  is not fixed and it does not belong to a compact manifold, thus homotopy group techniques say that  $L$  ( and  $Q'$ ) in (6) is an arbitrary real numbers. Furthermore, for  $|L| < |Q|$ , from Eq. (5), a decreasing of the lower bound of  $F$  is obtained with respect to the case  $\mathbf{c} = \mathbf{0}$ <sup>26</sup>. This effect is due to the interaction term  $-2F_{ik}H_{ik}$  in Eq. (4). Its contribution could be so important to completely compensate the energy contributions  $F_{\mathbf{c}}$  and  $F_{\mathbf{n}}$ . Such a situation may occur because all contributions in Eq. (4) are of the same order. Then, one can prove<sup>28</sup> the existence of inhomogeneous states with total energy smaller than in the Skyrme-Faddeev reduction (i.e.  $\mathbf{c} = \mathbf{0}$ ,  $\rho = const$ ), if the knot is contained in a region of typical size  $\sqrt{2} > R \sim 1/\rho$  and the amplitude of the momentum field  $|\mathbf{c}| \sim c_0$  is bounded by  $\alpha/R_0 < c_0 < 1/R_0$ , where  $R_0$  can be figured out as the "thickness" of the filamentary structure of  $\mathbf{c}$ , or  $\mathbf{n}$ , and  $\alpha = (R_0/R)^2 \ll 1$  is the packing parameter. Thus, by (5) one can evaluate the decrease of energy, by a negative (condensation energy) contribution  $\Delta F \sim (64\pi^2/Q^{1/4})\alpha$ .

In the optimum case of great values  $\mathbf{c}$  the self-dual state with  $F_{\mathbf{n}} = F_{\mathbf{c}}$  may be considered as the ground state with  $F_{min} = 0$ . It is characterized by the dense packing of filaments in knots<sup>36</sup> and by the condition  $\mathbf{a} - \mathbf{c} = \mathbf{A} = 0$  when the matrix

$$K_{\alpha\beta} = Q \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \quad (7)$$

Indeed, let us assume that the amplitude of density  $\rho^2 \sim R^{-1}$  is large enough and a knot size  $R$  is so small that the packing degree of filaments in the knot<sup>36</sup>  $\alpha = \xi/R \lesssim 1$ . The correlation length  $\xi$  having the order of the lattice constant is determined the filament thickness. The decrease of our tuning parameter  $\alpha$  with the decrease of  $\rho^2$  is accompanied by the transition to a state, in which among planar field projections the field configurations in the form of closed one-dimensional distributions are most preferable. In accordance with Ginzburg's proposal, we will call this phase state with the spontaneous diamagnetism a toroid state<sup>37,38</sup>. The phase state with zero value of the total magnetic moment  $\sum_i \mathbf{m}_i$ , in which current order is characterized by a polar vector  $\mathbf{T}$ , that changes sign under time reversal, can be represented by the ordering of toroid moments. The toroid moment  $\mathbf{T} = \frac{1}{2} \sum_i [\mathbf{r}_i \times \mathbf{m}_i]$  can be determined in the following way<sup>37,38</sup>

$$\mathbf{a} = \text{curl curl } \mathbf{T}. \quad (8)$$

Using this equation and the identity<sup>39</sup>

$$(\text{curl } \mathbf{a})_i = \frac{1}{2} \varepsilon_{ikl} \mathbf{n} [\partial_k \mathbf{n} \times \partial_l \mathbf{n}] \quad (9)$$

one can express the toroid moment via the degree of chirality  $\mathbf{n} \cdot [\partial_k \mathbf{n} \times \partial_l \mathbf{n}]$  in the form:

$$T_i(\mathbf{r}) = \frac{1}{16\pi^2} \int d^3r' \left[ \frac{(\mathbf{r} - \mathbf{r}')_k}{|\mathbf{r} - \mathbf{r}'|^3} \int d^3r'' \frac{\mathbf{n} \cdot [\partial_k \mathbf{n} \times \partial_i \mathbf{n}]}{|\mathbf{r}' - \mathbf{r}''|} \right]. \quad (10)$$

We see that toroid moment  $\mathbf{T}$  is given by the Biot-Savart law as well as by the factor which is determined by the Coulomb Green function and by the field strength  $H_{ik}$  of the Hopf invariant density. The vector  $\mathbf{T}$  characterizes the distribution of the poloidal component of the current on the torus<sup>38</sup>. The magnetic flux of this current is confined to the interior of the torus. If the degree  $\mathbf{n} \cdot [\partial_k \mathbf{n} \times \partial_i \mathbf{n}] = \varepsilon_{ki} \delta(\mathbf{r})$  of the  $\mathbf{n}$ -field noncollinearity is localized at a point the toroid moment has the form  $T_i(\mathbf{r}) = (8\pi)^{-1} \varepsilon_{ki} x_k / r$ . The toroid moment is perpendicular to the plane, where the central loop of the torus is situated, and belongs to the perpendicular plane that intersects this loop.

The significant constraint of the scale of toroid configuration appears from the analysis of stability of such field distributions. As it has been shown in Ref.<sup>40</sup>, that these configurations are stable with respect to  $s$ -wave perturbations if the characteristic scale of linked distribution is small enough, i.e. it is of the order of several lattice constants. Therefore, to describe topological states we have to use a lattice theory. Besides, in addition, the main contributions of field configurations correspond usually to small values of topological numbers  $Q$ ,  $L$ . Keeping all this in mind we can conclude that the Hopf links of loop pairs (see Fig. 1), that after being projected on the plane, provide dimer distributions of vortex pairs which are the most significant configurations. To deal with states without the time inversion symmetry breaking, we should take into consideration totally neutral dimer configurations with antitoroidal ordering of the toroid moments.

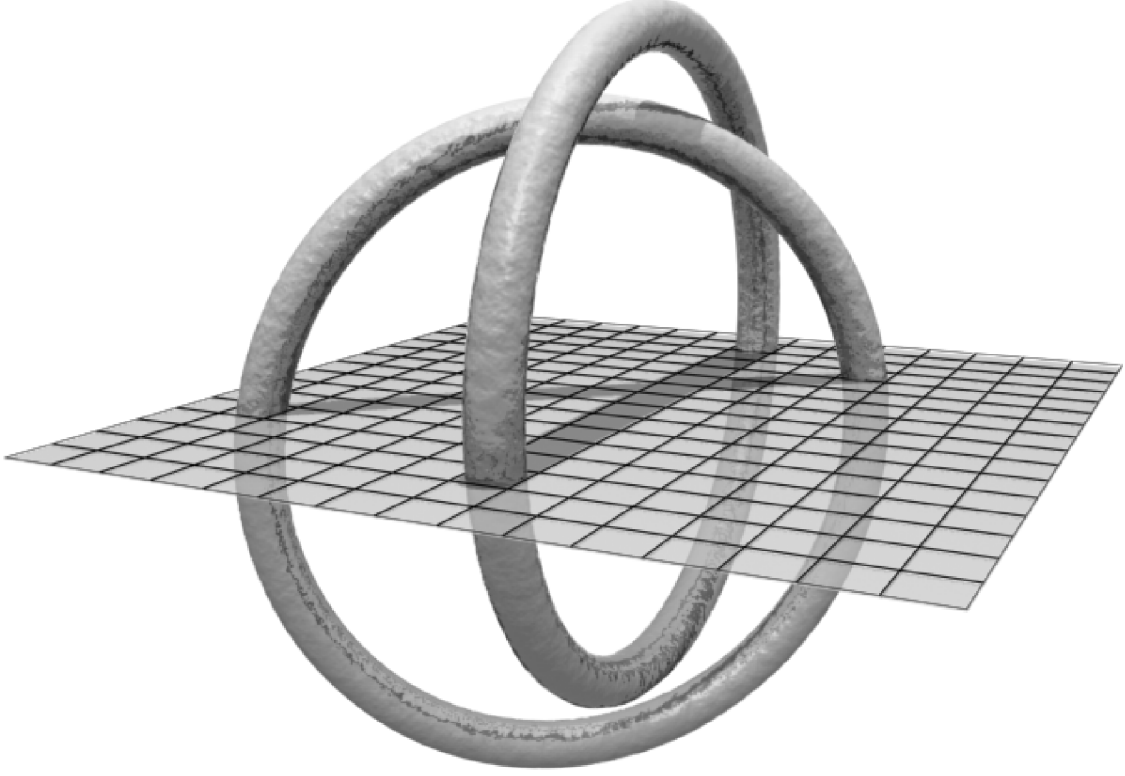


Figure 1: (Color online). The Hopf pair with  $Q = 1$ . Their projection into the plane presents a system of two dimers, characterized by the respective orientation of toroid moments.

### III. DISCRETE EQUATIONS OF EXCLUSION STATISTICS

Let us consider a system, which contains a set,  $\{N_a\}$  of particles with types  $a$ . The collective index  $a = (\alpha, i)$  contains the index  $\alpha$  for denoting internal degrees of freedom and index  $i$  enumerates rapidities of particles. If we fix the variables of all particles, except the  $a$ -th one, the  $N$ -particle wave function can be expressed via the one-particle function of the  $a$ -th particle. Let  $D_a$  be the dimension of such a basis. Then the rate of changing the number of vacant states due to adding  $N_b$  particles determines<sup>7</sup> the matrix  $g_{ab}$  of statistical interaction in the following way

$$\frac{\partial D_a}{\partial N_b} = -g_{ab}. \quad (11)$$

Assuming that the matrix  $g_{ab}$  does not depend on the set of numbers  $\{N_a\}$ , we have the solution of the Eq. (11):

$$D_a = - \sum_b g_{ab} N_b + D_a^0. \quad (12)$$



The Eq. (12) contains the number of particles  $N_b$ , added to the system, and the number of vacant states  $D_a^0$  of the  $a$ -th type in the initial state without particles. The number of holes  $D_a$  determines the statistical weight as follows

$$W = \prod_a \frac{(N_a + D_a - 1 + \sum_b g_{ab} \delta_{ab})!}{(N_a)!(D_a - 1 + \sum_b g_{ab} \delta_{ab})!}. \quad (13)$$

In the cases  $g_{ab} = 0$  and  $g_{ab} = \delta_{ab}$  the Eq. (13) yields well-known statistical weights of Bose and Fermi particles.

The statistical weight  $W$  allows to find the entropy  $S = \ln W$  and thermodynamical functions. The free energy in the equilibrium state

$$F = -T \sum_a D_a^0 \ln(1 + w_a^{-1}) \quad (14)$$

is determined by the function  $w_a$ , which can be found from the equation

$$(1 + w_a) \prod_b (1 + w_b^{-1})^{-g_{ab}} = e^{(\epsilon_a^0 - \mu_a)/T} \quad (15)$$

The variable  $w_a$  can be expressed via so-called pseudo-energies  $\epsilon = T \ln(D_a/N_a)$  by means of the parametrization  $w_a = e^{\epsilon_a/T}$ . In Eq. (15),  $T$ ,  $\mu_a$  and  $\epsilon_a^0$  are the temperature, the chemical potential and the bare energy of quasiparticles of the type  $a$ .

We consider the solution of the equation (15) in the limit  $T \gg \epsilon_a^0 - \mu_a$ . In this case the Eq. (15) can be written down in the form

$$w_a = \prod_b (1 + w_b^{-1})^{N_{ab}} \quad (16)$$

which is typical for the thermodynamic Bethe ansatz. Here  $N_{ab} = g_{ab} - \delta_{ab}$ .

Below we will be interested in the case of ideal statistics<sup>9,12,17</sup>, when phases of the scattering matrix, being functions of rapidities, have the structure of step functions. In this case the integral equation (16) transforms into the algebraic transcendental equation. The matrix  $N_{ab}$  can be expressed<sup>41,42,43,44</sup> via the incidence matrix  $G_{ab} = \delta_{a+1,b} + \delta_{a,b+1}$  of Lie algebra with the help of the identity  $N = G(2 - G)^{-1}$ . The matrix  $2 - G$  is the Cartan matrix of the graph  $A_{k+1}/Z_2$ <sup>41</sup>. Using this identity in Eq. (16) and replacing  $w_a = d_a^2 - 1$  it is easy to see that the Eq. (16) has the form

$$d_a^2 = 1 + \prod_{j=1, b=2j}^{[k/2]} d_b^{G_{ab}} = \begin{cases} 1 + d_2, & a = 1, \\ 1 + d_{a-1}d_{a+1}, & a = 2, \dots, [k/2] - 1, \\ 1 + d_{[k/2]-1}d_{[k/2]}, & a = [k/2]. \end{cases} \quad (17)$$

Here index  $a$  is connected with the value of the spin  $j$  by the relation  $a = 2j$ , and the upper limit of the product fixes the Jones-Wenzl projector<sup>22,23</sup>. We will show in Appendix that the Eq. (17) is presented in fact the special limit of the Hirota equation<sup>45</sup>.

The distribution function

$$n_a = \frac{1}{d_a^2} = \frac{1}{w_a + 1} = \frac{1}{e^{\epsilon_a/T} + 1} \quad (18)$$

in our case coincides with the probability  $p(a\bar{a} \rightarrow 0)$ <sup>46</sup> of annihilation of a particle-antiparticle pair in the system of two linked loops of world lines which describe the process of annihilation of two pairs.

We can find the solution of the Eq. (17) taking into account the appropriate boundary conditions by comparing it with the identity

$$[a]_q^2 - 1 = [a + 1]_q [a - 1]_q. \quad (19)$$

Here  $[a]_q = (q^a - q^{-a})/(q - q^{-1})$ ,  $q = e^{i\pi/(k+2)}$  is the deformation parameter of the  $SU(2)_k$  Chern-Simons theory. Identifying  $d_a = [a + 1]_q$ , we can see that solutions of the Eq. (17) are quantum dimensions<sup>46,47,48,49,50</sup>

$$d_a = \frac{\sin[\pi(a + 1)/(k + 2)]}{\sin[\pi/(k + 2)]}, \quad (20)$$

which are expressed via the Chebyshev polynomials of the second kind,  $U_m = \sin[(m + 1)\theta]/\sin \theta$  with specification  $\theta = \pi/(k + 2)$  for  $A_{k+1}$  algebra. In the limit  $k \gg 1$ ,  $d_{a=2j}$  equals  $2j + 1$ . The meaning of the quantum dimension is as follows. The quantum dimension  $d_a$  determines the rate  $d_a^N$  at which the dimension of the topological Hilbert space grows after particles are added.

We pay our attention to the fact that the Eq. (17) is a fermionic representation<sup>51,52,53</sup> of the recursion relation for the Chebyshev polynomials of the second kind. The bosonic representation of recursion relations has the form  $U_{m+1}(x) + U_{m-1}(x) - 2xU_m(x) = 0$ . From this point of view, we can call the Eq. (26) (see Appendix) the anyon representation of recursion relations for the Chebyshev polynomials of the second kind. The roots of the Chebyshev polynomials, being the eigen values of the matrix  $G$ , are equal to

$$x_{m,k} = q^{m+1} + q^{-(m+1)} = 2 \cos \left( \frac{(m + 1)\pi}{k + 2} \right). \quad (21)$$

The greatest eigen value  $x_{0,k}$  of the incidence matrix  $G$  is given by the Baraha numbers

$$d = 2 \cos[\pi/(k + 2)]. \quad (22)$$

In particular for the special value  $k = 3$ , we have the golden ratio  $d = (1 + \sqrt{5})/2$  which is the solution of the algebraic equation  $d^2 = d + 1$ . For  $k = 2$ , the Baraha number  $d$  and the quantum dimension  $d_{2j=1} = \sqrt{2}$  coincide.

To clarify the meaning of the  $d$ 's one should emphasize that (i) the numbers  $d$  determine eigen values of Wilson operator for the contractible unknotted loop<sup>22,23</sup>. (ii) For special values of the parameter  $d = q + q^{-1}$ , the generators  $B_i = I - qe_i$  satisfy the relations of a braid group under the condition, that the generators  $e_i$  satisfy the relation  $e_i^2 = de_i$  of the Temperley-Lieb algebra. (iii) The values of the parameter  $d$  are nontrivial restriction, which leads to the finite-dimensional Hilbert spaces. (iv) The wavefunction  $\Psi$ , defined on the one-dimensional manifold, which is a joining up of the arbitrary tangle  $\alpha$  and the Wilson loop  $\bigcirc$ , i.e.  $\Psi(\alpha \cup \bigcirc)$ , equals  $d\Psi(\alpha)$ <sup>23</sup>. Thus the parameter  $d$  has the meaning of the weight of the contractible unknotted Wilson loop, and  $d^2$  acquires the meaning of fugacity<sup>54</sup>. (v) Besides, it turns out, that for the mentioned values of  $d$ , the theory is unitary.

Summarizing one can say that the points of intersection of braid statistics and statistics with the generalized exclusion principle are the set of the Baraha points  $d = 2 \cos[\pi/(k+2)]$ , where the processes of braiding and fusion of string manifolds are self-consistently united.

#### IV. DISCUSSION

The problem of the coexistence<sup>24</sup> of locality and braiding can be solved by constructing Hamiltonians  $H = \sum_i H_i$  of the Rokhsar-Kivelson (RK) type<sup>31</sup>. Each term  $H_i = Q_i^+ Q_i$  in the sum with  $Q = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$  acts at the RK point as a projector built by dimer configurations. If we locate the dimers on the opposite links of plaquettes we will encounter contradiction due to spatial separation of the braiding phenomena. The best way to solve this problem is the distribution of dimer states between odd and even sites of the lattice<sup>33</sup>. Another way corresponds to the use of the Fischer lattice<sup>55</sup>. The checkerboard distribution of linked dimer degrees of freedom with defects in the order of effective fluxes (Fig. 2) may be one of the possible ways to solve the problem.

The realistic candidate for the Hamiltonian with such a type of the ground state is the Hamiltonian  $H_i$  which contains the Temperley-Lieb generators  $e_i$  in the form of the projectors

$$H_i = \frac{1}{d} e_i \quad (23)$$

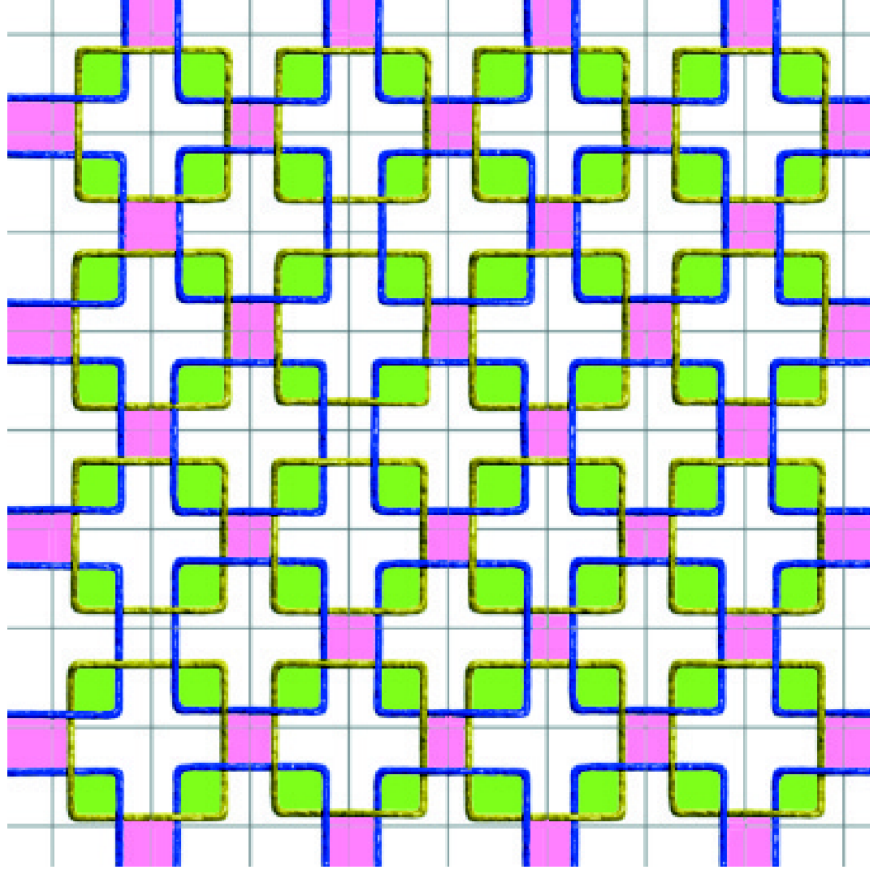


Figure 2: (Color online). A planar representation of the net. Shaded regions show distributions of the fluxes.

to the singlet states. Obviously  $H_i^2 = H_i$  due to the Temperley-Lieb commutation relation  $e_i^2 = de_i$ . The  $d$ 's here are the Baraha numbers from the second section. Because of the rank-level symmetry, i.e.  $SU(N)_k = SU(k)_N$ , and the argument based on small values of integers, the  $SU(2)_2 \times \overline{SU}(2)_2$  theory is a good candidate. The matrix of the  $6j$ -symbols<sup>46</sup> in this case is equal to  $\pm \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$  and the braid operator is  $\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$ . Another important model with computational universal rules is based<sup>33</sup> on a golden chain built by  $SU(2)_3 \times \overline{SU}(2)_3$  Fibonacci anyons. To construct 2D nets consisting of Fibonacci golden chains we should employ the Kirby calculus. This calculus is based on the application of Hopf links and is widely used in the theory of 3-manifolds.

In summary, we found the quantum dimensions as exact solutions of discrete equations encoding braiding and fusion processes. By means of projection of knotted field configurations

we proposed the quantum dimer models which incorporated braiding properties of one-dimensional topology.

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## V. APPENDIX

Let us show that the Eq. (17) is a particular case of the Hirota equation<sup>45</sup>

$$T_t^a(u+1)T_t^a(u-1) - T_{t+1}^a(u)T_{t-1}^a(u) = T_t^{a+1}(u)T_t^{a-1}(u). \quad (24)$$

Here  $a$  is the index of the algebra  $A_{k+1}$ ,  $t$  is the discrete time and  $u$  is the discrete values of rapidities. The functions  $T_t^a(u)$  are the eigen values of the transfer-matrix<sup>45</sup>. For the gauged functions  $Y_t^a(u) = T_{t+1}^a(u)T_{t-1}^a(u) / (T_t^{a+1}(u)T_t^{a-1}(u))$  the following equation

$$Y_t^a(u+1)Y_t^a(u-1) = \frac{(1+Y_{t+1}^a(u))(1+Y_{t-1}^a(u))}{(1+(Y_t^{a+1}(u))^{-1})(1+(Y_t^{a-1}(u))^{-1})} \quad (25)$$

is valid. In the  $A_1$ -algebra case the function  $Y_t^1(u) \equiv Y_t(u)$  satisfies the equation

$$Y_t(u+1)Y_t(u-1) = (1+Y_{t+1}(u))(1+Y_{t-1}(u)). \quad (26)$$

Putting  $Y_t = b_t^2 - 1$  in this equation, in the limit  $u \gg 1$  we get the equation  $b_t^2 = 1 + b_{t+1}b_{t-1}$  which coincides with (17). We see that the function  $Y_t$  equals  $w_t$ .

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